FOUR PAPERS ON PROBABILITY

ON THE DISTRIBUTION OF VALUES OF SUMS OF RANDOM VARIABLES

By

K. L. Chung and W. H. J. Fuchs

PROBABILITY LIMIT THEOREMS ASSUMING ONLY THE FIRST MOMENT I

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ON THE DISTRIBUTION OF VALUES OF SUMS OF RANDOM VARIABLES

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1. Notation and Summary. $X_1, X_2, \ldots$ are independent, identically distributed random vectors in Euclidean space of k dimensions ($k=1,2,3$). The distribution function of X_1 is $F(x)$ (or $F(x,y)$ or $F(x,y,z)$ as $k=1,2$, or 3), the characteristic function of X_1 is $\phi(u)$ ($\phi(u,v), \phi(u,v,w)$). $S_n = X_1+X_2 \ldots +X_n$. $|Y|$ denotes the maximum of the absolute values of the components of the vector Y.

The value b is possible, if to every $\varepsilon > 0$ there is an n such that $\Pr\{|S_n - b| < \varepsilon\} > 0$.

The value b is recurrent if for every $\varepsilon > 0$

$\Pr\{|S_n - b| < \varepsilon$ for an infinity of $n\} = 1$.

Theorem 1. Either no value is recurrent or all possible values are recurrent.

Theorem 2. There are recurrent values, if and only if for $h > 0$

$$\sum_{n=1}^{\infty} \Pr\{|S_n| < h\} = \infty \qquad (1.1).$$

Theorem 3. There are recurrent values, if for some $\alpha > 0$

$$\overline{\lim_{\rho \to 1-0}} \int_{-\alpha}^{\alpha} \cdots \int \frac{du \ldots}{1-\rho\phi(u,\ldots)} = \infty^{**} . \qquad (1.2)$$

(The number of integrations equals the dimension k of the vectors X_n.)

If for some $\alpha > 0$ and $0 < \rho < 1$

$$\int_{-\alpha}^{\alpha} \cdots \int \frac{du \ldots}{1-\rho\phi(u,\ldots)} < K < \infty \qquad (1.3)$$

then there are no recurrent values.

Since the real part of the integrand in (1.2) is positive (see § 3), we obtain by an application of Fatou's Lemma the following

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*In connection with an ONR project.

**The limit of the left hand side exists, but this is not needed here.

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Corollary. A sufficient condition for the existence of recurrent values is

$$\int_{-\alpha}^{\alpha}\cdots\int \frac{du\ldots}{1-\phi(u,\ldots)} = \infty$$

for some $\alpha > 0$.

Theorem 4. If $X_1, X_2 ..$ are independent, identically distributed real-valued random variables whose distribution function $F(x)$ satisfies

$$\int_{-\infty}^{\infty} |x| dF < \infty, \quad \int_{-\infty}^{\infty} x\, dF = 0 \tag{1.4}$$

then every real number is recurrent, unless all values assumed by X_1 are integral multiples of a fixed number. In this case all (possible and) recurrent values are given by $b=n\lambda$ $(n=0, \pm 1, \pm 2 \ldots)$.

In particular, under the hypotheses of Theorem 4

$$\overline{\lim}\ S_n = \infty$$

with probability one. This result is interesting in view of the fact that W. Feller* proved the existence of a distribution satisfying (1.4) and such that for arbitrarily small $\eta > 0$

$$\Pr\{S_n < -n(\log n)^{-\eta}\} \longrightarrow 1$$

as $n \longrightarrow \infty$.

Theorem 5. If $X_1, X_2 \ldots$ are independent, identically distributed vectors in two-dimensional Euclidean space whose distribution function $F(x,y)$ satisfies

$$\iint_{-\infty}^{\infty} x dF = \iint_{-\infty}^{\infty} y\, dF = 0$$

$$\iint_{-\infty}^{\infty} (x^2+y^2) dF < \infty, \tag{1.5}$$

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*Note on the law of large numbers and 'fair' games. Ann. Math. Statistics 16(1945)PP301-304.

then every possible value is recurrent.

Theorem 6. If $X_1, X_2, \ldots$ are independent, identically distributed random vectors with a genuinely three-dimensional distribution, then no value is recurrent.*

Acknowledgement. The simple proofs of Theorems 1 and 2 are due to Professor W. Feller. We are also indebted to him for much valuable advice on other points.

2. Proof of Theorem 1. α) Obviously every recurrent value is also possible.

β) If b is a recurrent value and c is a possible value, then $b-c$ is recurrent: Suppose the contrary. Then for some $\varepsilon > 0$

$$q = \Pr\{|S_n-(b-c)| < 2\varepsilon \text{ for a finite number of indices } n \text{ only}\} > 0.$$

Since c is possible there is an index k such that $p=\Pr\{|S_k-c|<\varepsilon\} > 0$. Then

$$\begin{aligned}&\Pr\{|S_n-b| < \varepsilon \text{ for a finite number of } n \text{ only}\}\\ &> \Pr\{|S_k-c| < \varepsilon,\ |S_{k+n}-S_k-(b-c)|<2\varepsilon \text{ for a finite number of } n \text{ only}\}\\ &> p \cdot q > 0, \qquad (2.1)\end{aligned}$$

since the distribution of $S_{k+n}-S_k=X_{k+1}+\ldots+X_{k+n}$ is the same as that of S_n and independent of S_k. But (2.1) contradicts the fact that b is recurrent.

γ) Lemma 1. The set T of all recurrent values forms a closed, additive group.

The definition of a recurrent value implies that T is closed. If b and c are recurrent, then by α) and β) $b-c$ is recurrent. This proves the group property.

Corollary. For a one-dimensional distribution there are the following three possibilities:

1. T is the empty set. 2. T is the set of all real numbers. 3. T is the set of all integral multiples of a number λ. For these are the only closed additive groups of real numbers.

δ) If T is not empty, then $0 \in T$, by γ). If c is any possible value, then $0-c=-c \in T$, by β). Hence $c \in T$, by β). This proves Theorem 1.

Proof of Theorem 2. α) If for any $h > 0$

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*Theorems 5 and 6 generalize results of Pólya, Ueber eine Aufgabe der Wahrscheinlichkeitsrechnung betreffend die Irrfahrt in Straßennetz 84(1921)pp.149-160.

$$\sum_{n=1}^{\infty} \Pr\{|S_n| < h\} < \infty, \tag{2.2}$$

then by the Borel-Cantelli Theorem $\Pr\{|S_n| < h$ for an infinity of $n\} = 0$. In particular 0 is not recurrent and so T is empty.

β) (1.1) implies

$$q(2h) = \Pr\{|S_n| \geq 2h \ (n=1,2,\ldots)\} = 0.$$

Let

$$r(h) = \Pr\{|S_n| < h \text{ for a finite number of } n \text{ only}\}.$$

Then

$$1 \geq r(h) \geq \sum_{k=1}^{\infty} \Pr\{|S_k| < h,\ |S_{k+n}-S_k| \geq 2h, \quad (n=1,2,\ldots)\}$$

$$= \sum_{k=1}^{\infty} \Pr\{|S_k| < h\}\, q(2h),$$

since $S_{k+n}-S_k$ is independent of S_k and has the same distribution as S_n. This contradicts (1.1) unless $q(2h) = 0$.

γ) Suppose now that (1.1) holds for every $h > 0$. Then

$$r(h) = \sum_{m>1/h} \sum_{k=1}^{\infty} \Pr\{|S_k| < h - \tfrac{1}{m},\ |S_{k+n}| \geq h(n=1,2,\ldots)\} + q(h)$$

$$\leq \sum_{m>1/h} \sum_{k=1}^{\infty} \Pr\{|S_k| < h - \tfrac{1}{m}\} \Pr\{|S_{k+n}-S_k| \geq \tfrac{1}{m} \ (n=1,2,\ldots)\}$$

$$= \sum_{m>1/h} \sum_{k=1}^{\infty} \Pr\{|S_k| < h - \tfrac{1}{m}\}\, q(\tfrac{1}{m}) = 0.$$

This concludes the proof of Theorem 2.

3. Proof of Theorem 3. We give the proof for the case of a two-dimensional distribution. The proofs for other dimensions differ only in trivial details.

Let the components of S_n be P_n, Q_n.

$$g(s,t) = \Pr\{|P_n| < s,\ |Q_n| < t\}$$

is a non-decreasing function of s and t. Hence

$$h^2 \Pr\{|S_n| < h\} = h^2 g(h,h) \geq \int_0^h ds \int_0^h dt\, g(s,t).$$

By a well-known formula the right hand side is equal to

$$\frac{1}{\pi^2} \iint_{-\infty}^{\infty} \frac{1-\cos hu}{u^2} \frac{1-\cos hv}{v^2} \phi^n(u,v)\,du\,dv$$

Now if $0 \leq \rho < 1$,

$$\frac{1}{\pi^2} \iint_{-\infty}^{\infty} \frac{1-\cos hu}{u^2} \frac{1-\cos hv}{v^2} \frac{du\,dv}{1-\rho\phi(u,v)}$$

$$\geq \frac{A^2(h)}{\pi^2} \iint_{-h^{-1}}^{h^{-1}} \frac{du\,dv}{1-\rho\phi(u,v)}$$

since $\dfrac{1-\cos hu}{u^2} \geq A(h) > 0$ for $|u| \leq h^{-1}$. Hence

$$\sum_{n=0}^{\infty} \rho^n \Pr\{|S_n| < h\} \geq \frac{A^2(h)}{\pi^2 h^2} \iint_{-h^{-1}}^{h^{-1}} \frac{du\,dv}{1-\rho\phi(u,v)}.$$

Notice that the integral is real-valued, since $\phi(-u,-v) = \overline{\phi(u,v)}$. Since $|\phi| \leq 1$,

$$R\frac{1}{1-\rho\phi} = \frac{1-\rho R\phi}{|1-\rho\phi|^2} \geq \frac{1-\rho}{|1-\rho\phi|^2} > 0.$$

If (1.2) is true for a certain α, we have for $h < \alpha^{-1}$

$$\lim_{\rho \to 1-0} \sum_{n=0}^{\infty} \rho^n \Pr\{|S_n| < h\} = \infty.$$

But (1.2) holds a fortiori if we increase α, hence the above is true for all h. Hence (1.1) is true and there are recurrent values by Theorem 2.

Now let $G_n(x,y)$ be the distribution function of S_n. Then

$$\Pr\{|S_n| < h\} = \iint_{-h}^{h} dG_n(x,y) \leq \frac{1}{A^2(h^{-1})} \iint_{-h}^{h} \frac{1-\cos h^{-1} x}{x^2} \frac{1-\cos h^{-1} y}{y^2} dG_n(x,y)$$

$$\leq \frac{1}{A^2(h^{-1})} \iint_{-\infty}^{\infty} \frac{1-\cos h^{-1} x}{x^2} \frac{1-\cos h^{-1} y}{y^2} dG_n(x,y)$$

$$= \frac{1}{4A^2(h^{-1})} \int_0^{h^{-1}} ds \int_0^{h^{-1}} dt \int_{-s}^{s} du \int_{-t}^{t} dv\, \phi^n(u,v).$$

Hence

$$\sum_{n=0}^{\infty} \rho^n \Pr\{|S_n| < h\} \leq \frac{1}{4A^2(h^{-1})} \int_0^{h^{-1}} ds \int_0^{h^{-1}} dt \int_{-s}^{s} \int_{-t}^{t} \frac{du\,dv}{1-\rho\phi(u,v)}$$

$$\leq \frac{h^{-2}}{4A^2(h^{-1})} \iint_{-h^{-1}}^{h^{-1}} \frac{du\,dv}{1-\rho\phi(u,v)}.$$

If (1.3) is true for a certain α, we have for $h^{-1} < \alpha$

$$\lim_{\rho \to 1-0} \sum_{n=0}^{\infty} \rho^n \Pr\{|S_n| < h\} < \infty.$$

But (1.3) holds a fortiori if we decrease α, hence the above is true for all h.

4. Theorem 4 is a consequence of the Corollary of Lemma 1 and of the slightly more general

Theorem 4a. If $X_1, X_2, \ldots$ are independent, identically distributed random variables whose distribution function $F(x)$ satisfies

$$\int_{-Y}^{Y} x \, dF(x) = o\,(1) \tag{4.1}$$

$$Y \int_{|x| > Y} dF(x) = o(1) \tag{4.2}$$

as $Y \longrightarrow \infty$, then recurrent values exist.

Proof. We notice first that

$$\int_{-Y}^{Y} x^2 dF(x) = o(Y) \tag{4.3}$$

as $Y \longrightarrow \infty$. For writing $F(x) - F(-x) = F^*(x)$,

$$\int_{-Y}^{Y} = \int_{0}^{Y} x^2 dF^*(x) = Y^2(1-F^*(Y))+2\int_{0}^{Y} x(1-F^*(x))dx$$

$$= o(Y)+2\int_{0}^{Y} o(1)dx = o(Y).$$

Now

$$1-\rho \leq R(1-\rho\phi(u)) = 1-\rho+\rho . R(1-\phi(u))$$

$$\leq 1-\rho+\int_{-\infty}^{\infty} (1-\cos x u)\, dF(x)$$

$$\leq 1-\rho+\frac{1}{2}\int_{-1/|u|}^{1/|u|} (xu)^2 \, dF(x) + 2\int_{|x| > 1/|u|} dF(x)$$

since $0 \leq 1 - \cos x u \leq \min\,(2, \frac{1}{2}\,(xu)^2)$. Hence, by (4.2) and (4.3)

$$R(1-\rho\phi) \leq 1-\rho+ o(u^2|u|^{-1})+o(|u|) = 1-\rho+ o(|u|)$$

as $u \longrightarrow 0$. Also

$$|I\rho\phi(u)| \le \left| \int_{-\infty}^{\infty} \sin x\, u\, dF \right| \le \left| \int_{|x| \le 1/|u|} xu\, dF(x) \right| + O\left(\int_{|x| \le 1/|u|} x^2u^2 dF \right)$$

$$+ \int_{|x| > 1/|u|} dF \quad ,$$

since $\sin x\,u = x\,u + O((xu)^2)$. Hence, using (4.1) and (4.3) $|I\rho\phi(u)| \le o(|u|)$. Given $\varepsilon > 0$ and $\alpha > 0$ there is an $\alpha_0 \le \alpha$ such that in $0 < u < \alpha_0$

$$R \frac{1}{1-\rho\phi(u)} = \frac{R(1-\rho\phi(u))}{(R(1-\rho\phi(u))^2 + (I\rho\phi(u))^2}$$

$$\ge \frac{1-\rho}{(1-\rho+\varepsilon u)^2+(\varepsilon u)^2} \ge \frac{1-\rho}{4(1-\rho)^2+4(\varepsilon u)^2}$$

and so, using (3.2)

$$\int_{-\alpha}^{\alpha} \frac{du}{1-\rho\phi(u)} = 2R \int_{0}^{\alpha} \ge 2R \int_{0}^{\alpha_0} \ge \frac{1}{2}\int_{0}^{\alpha_0} \frac{1-\rho}{(1-\rho)^2+(\varepsilon u)^2}\, du,$$

as $\rho \longrightarrow 1$ the last integral tends to $\frac{1}{2} \int_{0}^{\infty} \frac{dv}{1+(\varepsilon v)^2} = \frac{\pi}{4\varepsilon}$.

Since ε is arbitrarily small, (1.2) must hold and the theorem is proved.

Remark. The conditions of Theorem 4 are not necessary and can be varied in several ways. E.g. we can replace (4.2) by

$$Y \int_{|x| > Y} dF(x) = O(1),$$

if at the same time (4.1) is strengthened to

$$\int_{-Y}^{Y} dF(x) = 0,$$

i.e. the distribution of X_1 is symmetrical. But some condition like (4.2) is necessary, even for symmetrical distributions, as the following example shows. Let $0 < \varepsilon < 1$,

$$F'(x) = F'_c(x) \begin{cases} = 0 & (|x| < 1) \\ = \frac{1}{2} c|x|^{-c-1} & (|x| > 1) \end{cases} \tag{4.4}$$

Then, for $u > 0$,

$$\begin{aligned} \phi(-u) = \phi(u) &= c \int_1^\infty x^{-c-1} \cos xu \, dx \\ &= 1 - c \int_1^\infty (1-\cos xu) x^{-c-1} dx \\ &= 1 - cu^c \int_u^\infty (1-\cos t) t^{-c-1} dt \\ &= 1 - cu^c \left(\int_0^\infty - \int_0^u \right) \\ &\leq 1 - Au^c. \end{aligned}$$

Hence

$$\int_{-\alpha}^{\alpha} \frac{du}{1-\rho\phi(u)} \leq 2 \int_0^{\alpha} \frac{du}{Au^c} = K < \infty,$$

so that no value is recurrent. Here

$$Y \int_{|x| > Y} dF = kY^{1-c} .$$

5. Proof of Theorem 5. Under the hypothesis

$$1-\phi(u,v) = \iint_{-\infty}^{\infty} (1-e^{i(xu+yv)})dF(x,y)$$

$$= O\left(\iint_{-\infty}^{\infty} (xu+yv)^2 dF\right).$$

Hence

$$|1-\phi(u,v)| \leq K \iint_{-\infty}^{\infty} (u^2+v^2)(x^2+y^2)dF = B(u^2+v^2).$$

$$\iint_{-\alpha}^{\alpha} \frac{dudv}{1-\phi(u,v)} \geq \frac{1}{B} \iint_{-\alpha}^{\alpha} \frac{dudv}{u^2+v^2} = \infty$$

and Theorem 5 follows from the Corollary of Theorem 3.

Remark. The condition (1.5) is not necessary, but it can not be relaxed very much:

Let

$$F(x,y) = \int_{-\infty}^{x}\int^{y} F'_c(\xi)F'_c(\eta)d\xi\, d\eta ,$$

where F'_c is defined by (4.4), but now with $1 < c \leq 2$. A calculation very similar to that in § 4 shows that near the origin

$$\phi(u,v) < (1-k|u|^c)(1-k|v|^c) \qquad (1 < c < 2)$$

$$< 1-K(|u|^c + |v|^c)$$

$$\phi(u,v) = (1+u^2 \log |u| + O(u^2))(1+v^2\log|v| + O(v^2)) \qquad (c = 2)$$

Hence, for c=2,

$$\iint_{-\alpha}^{\alpha}\frac{du\, dv}{1-\phi(u,v)} = 8 \int_0^{\alpha} du \int_0^{u} \frac{dv}{1-\phi(u,v)}$$

$$\geq A \int_0^{\alpha} du \int_0^{u} \frac{dv}{v^2 \log(1/v)} = A \int_0^{\alpha} \frac{du}{u \log (1/u)} = \infty$$

and there are recurrent values although

$$\iint (x^2+y^2)\, dF(x,y) = \infty .$$

If $1 < c < 2$, then

$$\iint_{-\alpha}^{\alpha} \frac{du\, dv}{1-\rho\phi(u,v)} \leq A \iint_{-\alpha}^{\alpha} \frac{du\, dv}{|u|^c+|v|^c} < 8A \int_0^{\alpha} du \int_0^{u} \frac{dv}{u^c} < \infty$$

and no value is recurrent. But

$$\iint (|x|^b+|y|^b) dF(x,y) = 2 \int_{-\infty}^{\infty} |x|^b F'_c(x) dx < \infty ,$$

if $b < c$. Hence the condition (1.5) can not be replaced by

$$\iint (|x|^b+|y|^b) dF(x,y) < \infty$$

with any $b < 2$.

6. Proof of Theorem 6. Our assumption is that there is no plane through the origin such that X_1 lies with probability one in this plane. (The distribution is genuinely three-dimensional). Hence there is a sufficiently large R so that

$$Q = \iiint_{-R}^{R} (ux+vy+wz)^2\, dF(x,y,z) > 0$$

for <u>all</u> u,v,w, with $u^2+v^2+w^2 \neq 0$. For the left hand side can be equal to 0 only if all possible values of X_1 with $|X_1| < R$ lie on the plane $ux+vy+wz = 0$ with probability one. Therefore Q is a positive definite form in u,v,w and hence

$$Q \geq M (u^2 +v^2 +w^2).$$

Choose $|u|,|v|,|w| < 1/R$. Then

$$R(1-\phi(u,v,w)) = 2 \iiint_{-\infty}^{\infty} \sin^2 \tfrac{1}{2} (ux+vy+wz)\, dF$$

$$\geq 2 \iiint_{-R}^{R} \geq 2\pi^{-2} Q > A(u^2+v^2+w^2),$$

since in $\max(|x|, |y|, |z|) < R$ $|ux+vy+wz| < 3$ and so

$$\left|\sin \frac{1}{2}(ux+vy+wz)\right| > \frac{2}{\pi} \cdot \frac{1}{2} |ux+vy+wz|.$$

Hence for $\alpha < 1/R$

$$\iiint_{-\alpha}^{\alpha} \frac{du\ dv\ dw}{1-\rho\phi(u,v,w)} \leq \iiint_{-\alpha}^{\alpha} \frac{du\ dv\ dw}{R(1-\phi)} \leq A \iiint_{-\alpha}^{\alpha} \frac{du\ dv\ dw}{u^2+v^2+w^2} < \infty .$$

Theorem 6 is now a consequence of Theorem 3.

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PROBABILITY LIMIT THEOREMS ASSUMING ONLY THE FIRST MOMENT I

By

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In this paper we consider sums of mutually independent, identically distributed random variables. An essential feature is that we assume only that the first moment is zero, or that both its positive and negative parts diverge. Part I here deals with lattice distributions. Perhaps the main results are Theorem 3.1 and Theorem 8. We hope to take up other cases later.

1. Let X be a random variable which assumes only integer values

$$P(X = k) = p_k$$

$$p_k \geq 0 \quad \sum p_k = 1^* \ .$$

A number is said to be a 'possible' value of an integer-valued random variable if its probability is positive. The possible values of X will be denoted by $u_i, i=1,2,\ldots;$ they may be finite or infinite in number. As usual $S_n = \sum_{k=1}^{n} X_k$ where the X_k are mutually independent, each having the same distribution as X.

To avoid minor complications, we shall assume that <u>every integer</u> c <u>is a</u> possible values of S_n if n is sufficiently large: $n \geq n_0(c)$. A set of necessary and sufficient conditions for this is the following:

(1) The u_i are not all of the same sign;

(2) The greatest common divisor of the set of differences $u_i - u_j, i,j=1,2,\ldots$ is equal to 1.

We shall call the following two sets of assumptions (0) and (∞) respectively:

(0) $$E(|X|) = \sum |k| p_k < \infty, \quad E(X) = \sum k p_k = 0$$

(∞) $$\frac{1}{2} E(|X|+X) = \sum_{k=0}^{\infty} k p_k = \infty, \ \frac{1}{2} E(|X|-X) = -\sum_{k=-\infty}^{0} k p_k = \infty.$$

Thus under (0) or (∞) (1) is always satisfied except in the trivial case $X \equiv 0$, which we exclude. If (2) is not satisfied, there are two possibilities: either all possible

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*In an unspecified summation the index runs from $-\infty$ to $+\infty$.

values of S_n are multiples of an integer > 1; or there exists an integer $m > 1$ and a complete residue class mod. m, $r_1, \ldots, r_m$ such that for a fixed j, all possible values of S_{nm+j}, $n=1,2,\cdots$, belong to the same residue class $r_j \pmod{m}$. It is not difficult to see how our statements and proofs should be modified for these cases.

In the following the letters a, a' denote arbitrary integers, A, A', B positive constants; ε, ε' arbitrarily small constants.

2. In this section we give some simple theorems on the bounds of $P(S_n=a)$. It is well known that under more restrictive assumptions more precise results can be obtained (see Gnedenko [1], van Kampen and Wintner [2], Esseen [3]).

THEOREM 1. *Under no assumptions about moments whatsoever,*

(1) $$P(S_n=a) \leq An^{-1/2}$$

where A *does not depend on* n *or* a. *If* $E(X^2) = \infty$, *then*

(2) $$\lim_{n\to\infty} n^{1/2}P(S_n=a) = 0.$$

Proof. The c.f.* of the d.f.† of X is

$$f(x) = \sum p_k e^{ikx}.$$

The c.f. of S_n if $(f(x))^n$, and we have

$$P(S_n = a) = \frac{1}{2\pi}\int_{-\pi}^{\pi} (f(x))^n e^{-iax}dx.$$

Suppose first that n is even: n=2m. We have

$$\left|\int_{-\pi}^{\pi} (f(x))^{2m}e^{-iax}dx\right| \leq \int_{-\pi}^{\pi} (|f(x)|^2)^m dx.$$

Now $|f(x)|^2$ is the c.f. of a symmetrical d.f., namely that of X + X' where X, X' are mutually independent and X' has the same distribution as −X. Hence we may write

$$|f(x)|^2 = \sum_{k=0}^{\infty} r_k \cos kx, \left(r_k \geq 0, \sum_{k=0}^{\infty} r_k = 1\right)$$

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*Characteristic function or Fourier-Stieltjes transform.

† Distribution function.

$$= 1 - 2 \sum_{k=0}^{\infty} r_k \sin^2 \frac{kx}{2}.$$

Suppose $r_\ell > 0$. If $x \leq \pi \ell^{-1}$,

$$2 \sum_{k=0}^{\infty} r_k \sin^2 \frac{kx}{2} \geq 2\Big(\sum_{k=0}^{\ell} r_k \big(\frac{kx}{\pi}\big)^2\Big) \geq \frac{2x^2}{\pi^2} \sum_{k=0}^{\ell} r_k k^2 > Ax^2$$

$$|f(x)|^2 < 1 - A\, x^2.$$

Hence

$$\int_{-\pi}^{\pi} |f(x)|^{2m} dx < \int_{-\pi\ell^{-1}}^{\pi\ell^{-1}} (1 - A\, x^2)^m dx + \int_{\pi\ell^{-1} < |x| \leq \pi} |f(x)|^{2m} dx.$$

It is known that if $\eta < |x| \leq \pi$, then $|f(x)| < 1 - \varepsilon(\eta)$. Therefore we have

$$\int_{-\pi}^{\pi} |f(x)|^{2m}\, dx \leq \int_{-\pi\ell^{-1}}^{\pi\ell^{-1}} e^{-A\, mx^2}\, dx + O((1-\varepsilon)^m).$$

(1) follows for even n. Noticing that $|f(x)|^n \leq |f(x)|^{n-1}$ we see that the same proof goes through for an odd n.

To prove (2), notice that the assumption $E(X^2) = \infty$ implies that $E((X+X')^2) = \infty$. Hence

$$\sum_{k=0}^{\infty} k^2 r_k = \infty,$$

and the A' in the foregoing can be taken arbitrarily large. q.e.d.

A lower bound for $P(S_n=a)$, under the assumption (0) or (∞), will be given in Theorem 2.2; we shall also show that our estimate is close to the best possible by exhibiting an example in Theorem 2.3. In one special case, however, we can prove a much stronger result, and this is Theorem 2.1.

THEOREM 2.1. _If the_ d.f. _of_ X _is symmetrical, and_ $E(|X|) < \infty$, _then_

(3) $$\lim_{n \to \infty} n\, P(S_n=a) = \infty.$$

Proof. Since $p_k = p_{-k}$, $f(x)$ is real. Since $f(0) = 1$ and $f(x)$ is continuous, there exists a $\delta > 0$ such that if $|x| < \delta$, $f(x) > 0$. We have

$$P(S_n=a) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (f(x))^n \cos ax\, dx,$$

$$\geq \frac{1}{4\pi} \int_{-\delta}^{\delta} (f(x))^n dx - O((1-\varepsilon)^n),$$

if $\delta < \frac{\pi}{3a}$. As in the proof of Theorem 1, we can write

$$1-f(x) = \sum_{k=0}^{\infty} 2r_k \sin^2 \frac{kx}{2}.$$

Since $\sum_{k=0}^{\infty} kr_k < \infty$,

$$\lim_{x \to 0} \frac{1}{|x|} \sum_{k=0}^{\infty} r_k \sin^2 \frac{kx}{2} = 0.$$

Hence given $\varepsilon > 0$, if $|x| < \delta_0(\varepsilon) < \delta$, $1 - f(x) \leq \varepsilon|x|$. Now

$$\int_{-\delta}^{\delta} (f(x))^n dx \geq \int_{-\delta_0}^{\delta_0} (1-\varepsilon|x|)^n dx = \frac{2-2(1-\delta_0\varepsilon)^{n+1}}{(n+1)\varepsilon}.$$

Since ε is arbitrary (3) follows.

THEOREM 2.2. <u>Under</u> (0) <u>or</u> (∞) <u>we have for every</u> $\varepsilon > 0$

(4) $$P(S_n=a) \geq (1-\varepsilon)^n$$

<u>if</u> $n \geq n(\varepsilon, a)$.

Proof. If the possible values of X are bounded, then $E(X^2) < \infty$. In this case it is well known and also easy to show that

$$\lim_{n \to \infty} n^{1/2} P(S_n=a) = A < \infty.$$

This is a much sharper result than (4). Hence we may assume that there are possible values of arbitrarily large magnitude.

Given $\varepsilon > 0$, there exist arbitrarily large z_1 and z_2 such that

$$\sum_{-z_1}^{z_2} p_k > 1-\varepsilon$$

and if $k > z_2, p_k < \varepsilon$. Now choose h' so large that

$$\left|\sum_{-h'}^{0} kp_k\right| > \sum_{0}^{z_2} kp_k$$

this is possible under (0) or (∞). Also there is a unique h such that

$$\sum_{0}^{h-1} kp_k < \left| \sum_{-h'}^{0} kp_k \right| \leq \sum_{0}^{h} kp_k.$$

Then $h > z_2$ and

$$hp_h \geq \sum_{-h'}^{h} kp_k = C \geq 0.$$

Define $p'_k = p_k$ if $k \neq h$, but

$$p'_h = p_h - Ch^{-1} \geq 0.$$

Then

$$b = \sum_{-h'}^{h} p'_k > 1 - \varepsilon - p_h > 1 - 2\varepsilon$$

$$\sum_{-h'}^{h} kp'_k = 0.$$

Now define a random variable X' as follows:

$$P(X' = k) = p'_k b^{-1} \qquad \text{if} \qquad -h' \leq k \leq h$$

$$= 0 \qquad \text{otherwise.}$$

Let $S'_n = \sum_{k=1}^{n} X'_k$ where the X'_k are mutually independent and each has the same distribution as X'. Since $p'_k \leq p_k$ for all k

$$P(S'_n = a) \geq b^{-n} P(S_n = a \mid -h' \leq X_k \leq h \text{ for } 1 \leq k \leq n).*$$

Hence

$$P(S_n = a) \geq P(S_n = a;\ -h' \leq X_k \leq h \text{ for } 1 \leq k \leq n)$$

$$\geq P(-h' \leq x \leq h)^n P(S_n = a \mid -h' \leq X_k \leq h \text{ for } 1 \leq k \leq n)$$

$$\geq (1-\varepsilon)^n b^n P(S'_n = a) \geq (1-\varepsilon)^n (1-2\varepsilon)^n A n^{-1/2}$$

where A depends on ε by definition of X'. This being true for all ε is equivalent to (4).

The idea of truncation in the preceding proof is due to Shizuo Kakutani. Theorem 2.2 was first proved under (0) by W. H. J. Fuchs using a result in Chung and Fuchs [4], namely

---

*$P(E|F)$ denotes the conditional probability of E under the hypothesis F.

(5) $$\sum_{n=1}^{\infty} P(S_n = a) = \infty .^*$$

A similar proof using (5) was also given by Kakutani. We sketch the latter proof as follows.

From (5) it follows by the Cauchy-Hadamard criterion

$$\overline{\lim_{n\to\infty}} \; (P(S_n=0))^{1/n} = 1.$$

Hence given $\varepsilon > 0$, there exists arbitrarily large m such that

$$P(S_m = 0) \geq (1-\varepsilon)^m.$$

Consequently for all integers $k > 0$,

$$P(S_{km} = 0) \geq (1-\varepsilon)^{km}.$$

We can also choose the aforesaid m so large that

$$\min_{m<v\leq 2m} P(S_v=a) = A' > 0.$$

Now fix m. If $n=(k+1)m+r$, $k > 0$, $\varepsilon < r < m$, we have

$$P(S_n=a) \geq P(S_{m+r} = a)P(S_{km} = 0)$$

$$\geq A'(1-\varepsilon)^{km} \geq A'(1-\varepsilon)^{-m}(1-\varepsilon)^n.$$

q.e.d.

THEOREM 2.3. We can construct an example satisfying (0) and such that for every given $B > 0$ there exists a sequence $\{n_v\}$ for which

$$P(S_{n_v} \geq 0) = O(n_v^{-B}).$$

Proof. Let $A_v, v=1,2,\ldots$ be a sequence of positive integers increasing to ∞ so fast that for every $\varepsilon > 0$,

$$A_v = O(A_{v+1}^{\varepsilon}).$$

Define

$$X = \begin{cases} -1 \text{ with prob. } \frac{1}{2} \\ A_v \text{ with prob. } 2^{-v}A_v^{-1} \end{cases} \qquad v=2,3,\ldots .$$

Then $E(X) = 0$. If k is sufficiently large

$$P(\max_{1\leq v\leq n} X_v > A_k) \leq n \sum_{v=k+1}^{\infty} \frac{1}{2^v A_v} \leq \frac{n}{A_{k+1}} .$$

---

* However, the assumption (∞) does not imply the truth of (5) (see [4]); thus the following proof does not hold under (∞).

Let

$$X^* = \begin{cases} X \text{ if } X \leq A_k \\ 0 \text{ if } X > A_k \end{cases}.$$

$S^*_n = \sum_{v=1}^{n} X^*_v$ where the X^*_v are mutually independent and each has the distribution of X^*.

We have

$$E(X^*) = -2^{-(k+1)}, \; E(X^{*2}) = O(A_k)$$

$$P(S_n \geq 0) | \max_{1\leq v\leq n} X_v \leq A_k) = P(S^*_n \geq 0)$$

$$\leq P(|S^*_n - E(S^*_n)| \geq |E(S^*_n)|).$$

Let m be an integer > 0, a routine computation shows that

$$E(|S^*_n - E(S^*_n)|^{2m} \leq K_m(\sum_{v=1}^{n} E(X^* + 2^{-(k+1)})^2)^m$$

$$\leq K'_m A_k^m n^m$$

where K_m, K'_m are two positive constants depending only on m. Hence

$$P(|S^*_n - E(S^*_n)| \geq |E(S^*_n)|) \leq O(A_k^m \, 2^{(k+1)2m} n^{-m}).$$

Now choose

$$n_k \sim A_{k+1}^{\varepsilon}$$

we have, by the property of the sequence A_k,

$$P(S_{n_k} \geq 0) \leq P(\max_{1\leq v\leq n_k} X_v > A_k) + P(S_{n_k} \geq 0 | \max_{1\leq v\leq n_k} X_v \leq A_k)$$

$$\leq O(n_k^{-1/\varepsilon+1} + n_k^{-m+\varepsilon}) = O(n_k^{-B})$$

by choice of ε and m.

Theorem 2.3 should be compared with a result due to Feller [5].

3. The theorems in § 2 were proved by fairly standard analytical methods. We are unable to prove the theorems in this section by similar methods, except in the case where the d.f. of X is symmetrical, i.e., the c.f. is a real-valued function. In this case we have as before

$$|P(S_n=a) - P(S_n=a')| \le \int_{-\delta}^{\delta} |\cos ax - \cos a'x|\; |f(x)|^n\, dx + O((1-\varepsilon)^n).$$

Choosing δ so small that $\cos ax > 0$, $f(x) > 0$, and $|\cos ax - \cos a'x| < \varepsilon' \cos ax$ for $|x| \le \delta$, we have

$$|P(S_n=a) - P(S_n=a')| \ge \varepsilon' \int_{-\delta}^{\delta} \cos ax (f(x))^n\, dx + O((1-\varepsilon)^n).$$

On account of Theorem 2.2 it follows that

$$P(S_n=a) - P(S_n=a') = o\,(P(S_n=a))$$

which is equivalent to Theorem 3.1 below. We have not been able to prove the theorem by this method when $f(x)$ is not real-valued. Another relevant remark is the following: if instead of the individual probabilities $P(S_n=a)$ we consider their sums, then it follows from a theorem due to Doeblin [11] on Markov chains that

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} P(S_k=a)}{\sum_{k=1}^{n} P(S_k=a')} = 1.$$

THEOREM 3.1. <u>Under</u> (0) <u>or</u> (∞)

$$\lim_{n\to\infty} \frac{P(S_n=a)}{P(S_n=a')} = 1.$$

Proof. For some $k, u_1-u_0, \cdots, u_k-u_0$ have g.c.d. 1. Thus there exist integers c_i' and c_i such that

$$a'-a = \sum_{i=1}^{k} c_i'(u_i-u_0) = \sum_{i=0}^{k} c_i u_i, \quad \sum_{i=0}^{k} c_i = 0.$$

Let $P(X=u_i) = q_i$. Corresponding to every representation of a in the form

(1) $$a = \sum_{i=0}^{\ell} n_i u_i, \qquad n_i \ge 0, \qquad \sum_{i=0}^{\ell} n_i = n$$

there is a realization of the value a by $X_1+\cdots+X_n$ with probability

(2) $$\frac{n!}{n_0!\cdots n_\ell!} \prod_{i=0}^{\ell} q_i^{n_i}$$

when n_i of the X's assume the value u_i. The total probability of a is thus

$$\sum \frac{n!}{n_0!\cdots n_\ell!} \prod_{i=0}^{\ell} q_i^{n_i}$$

where the sum runs over all representations (1). Now write this sum as

(3) $$\sum = \sum_1 + \sum_2$$

where in $\sum_1$ the conditions

$$|n_i - nq_i| < \varepsilon n, \qquad 0 \leq i \leq \ell$$

are satisfied, while $\sum_2$ is the rest.

Consider the event $X = u_i$ with probability q_i; n_i is the number of its occurrences in n mutually independent, identical trials. It is well known that the probability that $|n_i - nq_i| > \varepsilon n$ is

$$O(e^{-\varepsilon'^2 n}).$$

Hence

(4) $$\sum_2 \leq (\ell+1)O(e^{-\varepsilon'^2 n}) = o(P(S_n=a))$$

by Theorem 2.2, for every $\varepsilon > 0$.

Now consider a representation (1) with $|n_i - nq_i| < \varepsilon n$ for $0 \leq i \leq \ell$. If ε is sufficiently small and n sufficiently large, we have $n_i > n(q_i - \varepsilon) > \varepsilon n > |c_i|$. Corresponding to every representation of a in the form (1) there is a representation of a' in the form

(5) $$a' = \sum_{i=0}^{k} (n_i + c_i)u_i + \sum_{i=k+1}^{\ell} n_i u_i = \sum_{i=0}^{\ell} n_i' u_i$$

where $|n_i' - nq_i| < 2\varepsilon n$. The ratio of two such corresponding probabilities is

$$= \frac{n_0!\cdots n_k!}{n_0'!\cdots n_k'!} q_0^{n_0'-n_0} \cdots q_k^{n_k'-n_k}.$$

If $m' > m$, $|m - nq| < \varepsilon n$, $|m' - nq| < 2\varepsilon n$, we have

$$\frac{m!}{m'!} q^{m'-m} \frac{(qn)(qn)\cdots(qn)}{m'(m'-1)\cdots(m+1)} n^{m-m'}$$

$$\leq \left(\frac{qn}{(q-2\varepsilon)n}\right)^{m'-m} n^{m-m'} = \left(\frac{1}{1-2\varepsilon}\right)^{m'-m} n^{m-m'}$$

$$\frac{m!}{m'!} q^{m'-m} \geq \left(\frac{1}{1+2\varepsilon}\right)^{m'-m} n^{m-m'}.$$

Since $\sum_{i=0}^{\ell} n_i' = \sum_{i=0}^{\ell} n_i$ it follows that

$$(1-\varepsilon')^C \leq \lambda \leq (1+\varepsilon')^C$$

where $C = \sum_{i=0}^{k} |c_i|$. Since ε' is arbitrarily small we have $\lim_{n\to\infty} \lambda = 1$.

Let us write the corresponding formulas (1) and (2) for a':

$$a' = \sum_{i=0}^{\ell} n_i' u_i, \qquad \sum_{i=0}^{\ell} n_i' = 0$$

(6) $$\sum{}' = \sum{}_1' + \sum{}_2'$$

where in $\sum_1'$ the condition $|n_i' - nq_i| < 2\varepsilon n$ are satisfied find $0 \leq i \leq \ell$. We have just proved that

$$\overline{\lim_{n\to\infty}} \frac{\sum_1}{\sum_1'} \leq 1.$$

Using (4) we conclude that

$$\overline{\lim_{n\to\infty}} \frac{P(S_n=a)}{P(S_n=a')} = 1.$$

Since a and a' are interchangeable we obtain Theorem 3.

THEOREM 3.2. <u>For those values of</u> n <u>for which</u>

(7) $$P(S_n=a) \geq n^{-B}$$

<u>for some fixed</u> $B > 0$, <u>we have for every</u> $\varepsilon > 0$

(8) $$|P(S_n=a) - P(S_n=a')| \leq P(S_n=a)An^{-1/2+\varepsilon}$$

<u>where</u> A <u>may depend on</u> a, a' <u>but not on</u> n.

Proof. In (3) we re-define $\sum_1$ to be the sum of those terms for which

$$|n_i - nq_i| < n^{1/2+\varepsilon}, \qquad 0 \leq i \leq \ell.$$

As before, let (5) correspond to (1), but now we assume n so large that $n^{1/2+\varepsilon} > |c_i|$, so that

$$|n_i' - nq_i| < 2n^{1/2+\varepsilon}.$$

We re-define $\sum_1'$ in (6) to be the sum of those terms for which this is true. By well known estimates on the binomial distribution we have

$$\sum_2 = O(e^{-An^{\varepsilon}}), \quad \sum_2' = O(e^{-An^{\varepsilon}}). \tag{9}$$

Now consider the difference of two corresponding probabilities (1) and (6)

$$d = \frac{n!}{n_0!\cdots n_\ell!} q_0^{n_0}\cdots q_\ell^{n_\ell}(1-\lambda).$$

If $m=nq+r$, $m'=nq+r'$ where $|r-r'| \le C$ and $|r| \le n^{1/2+\varepsilon}$, $|r'| \le 2n^{1/2+\varepsilon}$, an easy application of Stirling's formula yields

$$\frac{m!}{m'!} q^{m'-m} = n^{r-r'}(1+O(n^{-1/2+3\varepsilon})), \qquad \lambda = 1+O(kn^{-1/2+3\varepsilon}).$$

Since $\sum_{i=0}^{k} (r_i - r_i') = 0$ we have

$$|d| \le \frac{n!}{n_0!\cdots n_\ell!} q_0^{n_0}\cdots q_\ell^{n_\ell} O(n^{-1/2+3\varepsilon}).$$

Hence

$$|P(S_n=a) - P(S_n=a')| \le O(\textstyle\sum_1 . n^{-1/2+3\varepsilon}) + \sum_2 + \sum_2'.$$

The first term on the right is $O(P(S_n=a).n^{-1/2+3\varepsilon})$, and the other two terms by (7) and (9) are of smaller order of magnitude. Thus (8) follows.

THEOREM 4. _Under_ (0) _or_ (∞)

$$\lim_{n\to\infty} \frac{P(S_n=a)}{P(S_{n+1}=a')} = 1.$$

Proof. For every representation of a in the form (1), there is a representation of $a+u_0$ in the following form

$$a+u_o=(n_o+1)u_o + \sum_{i=1}^{\ell} n_i u_i.$$

The corresponding probability is

$$\frac{(n+1)!}{(n_o+1)!n_1!\cdots n_\ell!} q_o^{n_o+1} q_1^{n_1}\cdots q_\ell^{n_\ell}.$$

The ratio of this to (2) is $(n_o+1)/(n+1)q_o$. If $|n_o-nq_o| < \epsilon n$, this ratio is between $1-\epsilon/q$ and $1+\epsilon/q_o$ as $n \longrightarrow \infty$. The range at values of n_o such that $|n_o-nq_o| > \epsilon n$ can be neglected as before. It follows exactly as in the proof of Theorem 3 that

$$\overline{\lim_{n\longrightarrow\infty}} \frac{P(S_n=a)}{P(S_{n+1}=a+u_o)} \leq 1.$$

By virtue of Theorem 3 this gives

$$\overline{\lim_{n\longrightarrow\infty}} \frac{P(S_n=a)}{P(S_{n+1}=a)} \leq 1.$$

Considering $a-u_o$ instead of $a+u_o$ in the above in a similar manner we arrive at

$$\overline{\lim_{n\longrightarrow\infty}} \frac{P(S_n=a)}{P(S_{n-1}=a)} \leq 1.$$

These last two relations combined are equivalent to Theorem 4.

We remark that Theorem 4 can be proved in the same way as sketched above for Theorem 3.1, when $f(x)$ is real-valued. It would also seem that we might be able to deduce Theorem 4 directly from Theorem 3.1, but a trivial argument gives only the following. Since

$$P(S_{n+1}=a)=\sum_{a'=-\infty}^{\infty} P(S_n=a')\,P(X=a-a')$$

$$\geq \sum_{a'=-A}^{A} P(S_n=a')P(X=a-a').$$

It follows easily, using Theorem 3.1, that

$$\underline{\lim_{n\longrightarrow\infty}} \frac{P(S_{n+1}=a)}{P(S_n=a)} \geq 1.$$

But the other half of the result seems difficult.

4. In this section we study the number of a-values in the sequence $S_1,\ldots,S_n$. A very special case has been treated more or less completely by Chung and Hunt [6]. More general cases, in which the existence of certain moments are assumed, have been considered by Feller [7] and Chung [8].* In this paper we are considering a more general situation and precise results are not hoped for at this moment. However, we shall prove the relevant Theorem 8 whose truth would perhaps be considered evident but whose proof, as far as we can make it, is by no means simple. Theorem 7 gives the true bounds within an ε power.

Define

$$Y_k = \begin{cases} 1 \text{ if } S_k = a \\ 0 \text{ if } S_k \neq a \end{cases}$$

$$E(Y_k) = P(S_k=a) = m_k$$

$$T_n = \sum_{k=1}^{n} Y_k$$

$$E(T_n) = M_n = \sum_{k=1}^{n} m_k$$

and similarly Y'_k, m'_k, T'_n, M'_n, for a'.

THEOREM 5. Under (0), for every $\varepsilon > 0$,

$$P(|T_n - T'_n| > M_n^{3/4+\varepsilon} \text{ i.o.**}) = 0.$$

Proof. By Theorem 3.1 and the fact that $M_n \longrightarrow \infty$ as $n \longrightarrow \infty$

$$E(|T_n-T'_n|^2) = E\Big(\sum Y_k^2 + \sum {Y'}_k^2 + \sum_{j\neq k} Y_j(Y_k-Y'_k) + \sum_{j\neq k} Y'_j(Y'_k-Y_k)\Big) \dagger$$

$$\ll \sum m_j + \sum m'_j + \sum m_j \sum_{k\neq j} |m_{j-k}-m'_{j-k}| + \sum m'_j \sum_{k\neq j} |m'_{j-k}-m_{j-k}| \ddagger$$

$$\ll M_n + M_n \sum |m_k - m'_k|.$$

*The results in [8] are stated for the number of crossings of the values a, but in the case of an integer-valued random variable they can be easily translated into the number of a-values. ** i.o. stands for 'infinitely often' or 'for infinitely many values of the index.'

† Henceforth in an unspecified summation the index runs from 1 to n. ‡ $u_n \ll v_n$ means $u_n = O(v_n)$

According to Theorem 3.2, the m_k in the last sum can be divided into two classes: either $m_k \leq k^{-2}$, the sum over such k being $O(1)$, or the estimate (8) holds. Hence

$$E(|T_n-T'_n|^2) \leq O(M_n) + O(M_n \sum m_k k^{-(1-\varepsilon)/2}).$$

By Hölder's inequality

$$\sum m_k k^{-(1-\varepsilon)/2} \leq (\sum m_k^{2/(1+2\varepsilon)})^{1/2+\varepsilon} (\sum (k^{-(1-\varepsilon)/2})^{2/(1-2\varepsilon)})^{1/2-\varepsilon}$$

$$\leq A(\sum m_k^{2/(1+2\varepsilon)})^{1/2+\varepsilon} \leq AM_n^{1/2+\varepsilon} .$$

By Chebychev inequality

$$P(|T_n-T'_n| > M_n^{3/4+\varepsilon}) \leq M_n^{-\varepsilon} . \tag{10}$$

Since $m_n \longrightarrow 0$ by Theorem 1, we can choose an increasing sequence n_k such that

$$M_{n_k} \sim k^{(1+\varepsilon)/\varepsilon} .$$

Now suppose that for some $n, n_k < n \leq n_{k+1}$ we have

$$|T_n-T'_n| > 2M_n^{3/4+\varepsilon} . \tag{11}$$

Let n be the smallest such integer, for which (11) is true, then either Y_n or Y'_n must be 1, hence $S_n=a$ or a'. We call this event E_n. According as $S_n=a$ or a', $T_{n_{k+1}}-T_n$ is the number of 0's or $(a-a')$'s in the sequence of partial sums of $X_{n+1},\ldots,X_{n_{k+1}}$. Let the event

$$|T_{n_{k+1}}-T'_{n_{k+1}}-(T_n-T'_n)| \leq M_{n_{k+1}}^{3/4+\varepsilon}$$

be denoted by $E_{n,n_{k+1}}$. By (10), if k is sufficiently large,

$$P(E_{n,n_{k+1}}|E_n) \geq 1 - M_{n_{k+1}}^{-\varepsilon} \geq \frac{1}{2} .$$

Further it is clear that

$$P(E_{n,n_{k+1}}|E'_{n_k}\cdots E'_{n-1}E_n)^* = P(E_{n,n_{k+1}}|E_n) \geq \frac{1}{2}$$

* If E,F are two events, E' denotes the negation of E, EF denotes the conjunction of E and F.

(this follows from the Markov property of the sequence S_n.) Now $E_{n,n_{k+1}}$ and $E'_{n_k}\dots E'_{n-1}E_n$ together imply

$$|T_{n_{k+1}}-T'_{n_{k+1}}| > M_{n_{k+1}}^{3/4+\varepsilon}.$$

Hence

$$P(|T_{n_{k+1}}-T'_{n_{k+1}}| > M_{n_{k+1}}^{3/4+\varepsilon}) \geq \sum_{n=n_k}^{n_{k+1}} P(E'_{n_k}\dots E'_{n-1}E_n)P(E_{n,n_{k+1}}|E'_{n_k}\dots E'_{n-1}E_n)$$

$$\geq \frac{1}{2}\sum_{n=n_k}^{n_{k+1}} P(E'_{n_k}\dots E'_{n-1}E_n) = \frac{1}{2}P(\max_{n_k\leq n\leq n_{k+1}} |T_n-T'_n|M_n^{-3/4-\varepsilon} > 2).$$

Thus by (10)

$$\sum_k P(\max_{n_k\leq n\leq n_{k+1}} |T_n-T'_n|M_n^{-3/4-\varepsilon} > 2) \leq 2\sum_k M_{n_{k+1}}^{-\varepsilon} < \infty.$$

It follows from the Borel-Cantelli lemma that

$$P(|T_n-T'_n| > 2M_n^{3/4+\varepsilon} \quad \text{i.o.}) = 0.$$

This is equivalent to the statement of Theorem 5.

The next theorem is a new type of limit theorem. The sequence of random variables $Y_1,Y_2,\dots$ does not obey the usual law of large numbers in the sense that constants A_n do not exist so that with probability 1,

$$\lim_{n\to\infty} \frac{Y_1+\dots+Y_n}{A_n} = 1.$$

By analogy with the situation for sums of independent random variables with finite first moments, we should expect to take A_n to be the M_n above. That this is not true is shown already in the simplest case of Bernoullian variables $X_1,X_2,\dots$ where each $X_k = \pm 1$ each with probability 1/2. In this case $m_k \sim A\,k^{-1/2}$. $M_n \sim 2An^{1/2}$, but the sum $Y_1+\dots+Y_n$ oscillates between $A'n^{1/2}(\log n)^{-1-\varepsilon}$ and $A''n^{1/2}(\log\log n)^{1/2}$ with probability 1 (see [6]). However we shall show in the next theorem that, in a certain sense, Y_k does behave like its expectation m_k, as follows.

THEOREM 6. Under (0),

$$P\left(\lim_{n\to\infty} \frac{1}{\log M_n} \sum_{k=1}^{n} \frac{Y_k}{M_k} = 1\right) = 1.$$

Notice that in this formula if we replace Y_k by m_k, the limit relation holds without the intervention of probability. If we regard $(Y_1+\dots+Y_n)/M_n$ as a sort of 'arithmetical average,' the quantity

$$\frac{1}{\log M_n} \sum_{k=1}^{n} \frac{Y_k}{M_k}$$

may be called a 'logarithmic average.' Evidently the existence of the mathematical average implies the existence (and equality therewith) of the logarithmic. The first instance of considering such an average in probability is due to P. Levy [9], p.270.

Proof of Theorem 6. We have

$$E\left(\sum \frac{Y_k}{M_k}\right) = \sum \frac{m_k}{M_k} = \log M_n + O(1).$$

Next

$$E\left(\left(\sum \frac{Y_k}{M_k}\right)^2\right) = \sum \frac{m_k}{M_k^2} + 2 \sum_{j<k} \frac{m_j m_{k-j}}{M_j M_k}$$

$$= O(1) + 2 \sum_{j=1}^{n} \frac{m_j}{M_j} \sum_{k=j+1}^{n} \frac{m_{k-j}}{M_k}.$$

Hence

$$0 \le E\left(\left(\sum \frac{Y_k}{M_k}\right)^2\right) - E^2\left(\sum \frac{Y_k}{M_k}\right)$$

$$\le O(1) + O\left(\sum \frac{m_k^2}{M_k^2}\right) + 2 \sum_{j=1}^{n} \frac{m_j}{M_j}\left(\sum_{k=j+1}^{n} \frac{m_{k-j}}{M_k} - \sum_{k=j+1}^{n} \frac{m_k}{M_k}\right)$$

$$\le O(\log M_n) + 2 \sum_{j=1}^{n} \frac{m_j}{M_j}\left\{\sum_{k=1}^{n-j} \frac{m_k}{M_{k+j}} - \sum_{k=j+1}^{n} \frac{m_k}{M_k}\right\}$$

$$\leq O(\log M_n) + 2 \sum_{j=1}^{n} \frac{m_j}{M_j} \sum_{k=1}^{j} \frac{m_k}{M_{k+j}}$$

$$\leq O(\log M_n) + 2 \sum_{j=1}^{n} \frac{m_j}{M_j} = O(\log M_n).$$

By Chebychev inequality

$$P(|\sum \frac{Y_k}{M_k} - \log M_n| > \varepsilon \log M_n) \leq O(\frac{1}{\log M_n}).$$

Choose an increasing sequence n_k such that

$$M_{n_k} \sim e^{k^2}.$$

By the Borel-Cantelli lemma,

$$P(\lim_{k \to \infty} \frac{1}{\log M_{n_k}} \sum_{i=1}^{n_k} \frac{Y_i}{M_i} = 1) = 1.$$

Now if $n_k \leq n \leq n_{k+1}$,

$$\frac{1}{\log M_{n_{k+1}}} \sum_{k=1}^{n_k} \frac{Y_i}{M_i} \leq \frac{1}{\log n} \sum_{i=1}^{n} \frac{Y_i}{M_i}$$

$$\leq \frac{1}{\log M_{n_k}} \sum_{i=1}^{n_{k+1}} \frac{Y_i}{M_i} .$$

Since $\log M_{n_{k+1}} / \log M_{n_k} \longrightarrow 1$ as $k \longrightarrow \infty$ the extreme sides of these inequalities $\longrightarrow 1$ with probability 1, by what has just been proved. Theorem 6 follows.

THEOREM 7. Under (0), for every $\varepsilon > 0$

$$P(M_n^{1-\varepsilon} < T_n < M_n^{1+\varepsilon} \text{ for all sufficiently large } n) = 1.$$

Proof. This is equivalent to the following two statements:

(1) $P(T_n > M_n^{1+\varepsilon} \text{ i.o.}) = 0$

(2) $P(T_n < M_n^{1-\varepsilon} \text{ i.o.}) = 0.$

The proof of (1) is similar to that of Theorem 5 and will be omitted. To prove (2), we choose $v=v(n)$ such that $M_v \sim M_n^{1-\varepsilon}$; this is possible because $m_n \longrightarrow 0$ and $M_n \uparrow \infty$. From Theorem 6 we have, with probability 1,

$$\lim_{n\longrightarrow} \frac{1}{\log M_n} \sum_{k=1}^{v} \frac{Y_k}{M_k} = 1-\varepsilon$$

$$\lim_{n\longrightarrow\infty} \frac{1}{\log M_n} \sum_{k=v+1}^{n} \frac{Y_k}{M_k} = \varepsilon.$$

Upon subtraction it follows that

$$\lim_{n\longrightarrow\infty} \frac{1}{\log M_n} \frac{T_n-T_v}{M_v} \geq \varepsilon$$

or

$$\lim_{n\longrightarrow\infty} \frac{T_n}{M_n^{1-\varepsilon}\log M_n} \geq \varepsilon.$$

This is equivalent to (2).

Remark. Part (2) of Theorem 7 would also have followed from a general theorem of Feller (Theorem 2 in [10]), but for the condition (13) there. To verify this condition (or rather a slightly weaker one) it would be sufficient to show that

$$M_{2n} \leq 2M_n.$$

We are unable to prove or disprove this relation.

THEOREM 8. *Under* (0)

$$P(\lim_{n\longrightarrow\infty} \frac{T_n}{T'_n} = 1) = 1.$$

Proof. This is an immediate consequence of Theorems 5 and 7. Actually we have even, for every $\varepsilon > 0$

$$P(\left|\frac{T_n-T'_n}{T_n}\right| > M_n^{-(1+\varepsilon)/4} \quad \text{i.o.}) = 0.$$

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REMARKS ON FLUCTUATIONS OF SUMS OF INDEPENDENT RANDOM VARIABLES

By

K. L. CHUNG and M. KAC*

1. Introduction. Let $X_1, X_2, \ldots$ be independent, identically distributed random variables and set

(1.1) $$s_k = X_1 + X_2 + \ldots + X_k.$$

If each X happens to assume integral multiples of some fundamental unit it is of interest to investigate statistical properties of the number Z_n of zeros in the sequence $s_1, s_2, \ldots, s_n$. In this case general results were obtained by Feller [1]. If the X's do not have the 'lattice' structure described above one cannot speak sensibly of the zeros and the problem must be reformulated. Perhaps the most natural analogue of Z_n is the number N_n of changes of sign in the sequence $s_1, s_2, \ldots, s_n$ (the number of times the luck changes, to use the terminology of the theory of games of chance).
Another analogue is the random variable $M_n(a)$ which represents the number of s_k's, $1 \le k \le n$, such that $|s_k| < a$.
For X's with mean 0 and finite third moment (obeying the central limit theorem) K. L. Chung [2] determined the limiting distribution of $N_n/n^{1/2}$.
The limiting distribution turned out to be the 'truncated normal' which except for a numerical factor (depending on the ratio of the first absolute moments of X to the standard deviation) is also the limiting distribution of $Z_n/n^{1/2}$ in the case the 'lattice' valued variables obey the central limit theorem.
If the X's have the 'lattice' structure and belong to the domain of attraction of a symmetric stable law with exponent $\alpha > 1$, the limiting distribution of $Z_n/n^{1-1/\alpha}$ has been determined by Feller [1]. If $\alpha < 1$, it is easy to show that $\lim_{n \to \infty} Z_n < \infty$ with probability 1.

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It thus becomes of interest to investigate the statistical properties of N_n and $M_n(a)$ in the case the central limit theorem is violated and the normal distribution replaced by a general, symmetric, stable law.

In order to simplify the calculations and thus bring out more clearly the rather curious phenomena encountered in the non-Gaussian case, we shall restrict ourselves to the case when all the X's have the same, symmetric, stable distribution i.e.,

(1.2) $$E\{\exp(i\xi X_j)\} = \exp(-|\xi|^{\alpha}), \quad 0 < \alpha \leq 2.$$

Our results can be summarized as follows:

1. If $1 < \alpha \leq 2$, the limiting distribution of

$$\frac{N_n}{n^{1-1/\alpha}} \quad \text{and} \quad \frac{1}{a}\,\frac{M_n(a)}{n^{1-1/\alpha}}$$

exist and can be explicitly calculated. Moreover, except for numerical factors, they are identical with the limiting distribution of $Z_n/n^{1-1/\alpha}$ found by Feller [1].

2. If $\alpha = 1$ (Cauchy distribution) the limiting distribution of

$$\frac{\pi}{2a}\,\frac{M_n(a)}{\log n}$$

is exponential i.e.,

(1.3) $$\lim_{n\to\infty} \Pr\left\{\frac{\pi}{2a}\,\frac{M_n(a)}{\log n} < x\right\} = \int_0^x e^{-y}dy$$

whereas

(1.4) $$\lim_{n\to\infty} \Pr\left\{\frac{N_n}{2\pi^2(\log n)^2} < x\right\} = \int_0^x e^{-y}\,dy.$$

3. If $\alpha < 1$, $M_n(a)$ is bounded with probability 1, whereas

(1.5) $$\lim_{n\to\infty} \Pr\left\{\frac{N_n}{2D(\alpha)\log n} < x\right\} = \int_0^x e^{-y}\,dy,$$

where the constant $D(\alpha)$, will be given later.

2. The case $1 < \alpha \leq 2$.

We first consider $M_n(a)$. Let

$$(2.1) \qquad V_a(x) = \begin{cases} 1, & |x| < a, \\ 0, & |x| \geq a, \end{cases}$$

and denote by $\rho_k(x)$ the density function of s_k. We have

$$(2.2) \qquad \rho_k(x) = \frac{1}{k^{1/\alpha}} \rho_1\left(\frac{x}{k^{1/\alpha}}\right) \equiv \frac{1}{k^{1/\alpha}} \rho\left(\frac{x}{k^{1/\alpha}}\right).$$

Now,

$$(2.3) \qquad M_n(a) = \sum_1^n V_a(s_k)$$

and

$$E\{M_n(a)\} = \sum_{k=1}^n \frac{1}{k^{1/\alpha}} \int_{-a}^{a} \rho\left(\frac{x}{k^{1/\alpha}}\right) dx =$$

$$2 \sum_{k=1}^n \int_0^{a/k^{1/\alpha}} \rho(x)dx \sim 2a\, \rho(0) \sum_{k=1}^n \frac{1}{k^{1/\alpha}} .$$

In other words

$$(2.4) \qquad \lim_{n \to \infty} E\left\{ \frac{M_n(a)}{2a\, \rho(0)\, n^{1-1/\alpha}} \right\} = \int_0^1 \frac{dt}{t^{1/\alpha}} .$$

To calculate the second moment $E\{M_n^2(a)\}$ we need

$$E\{V_a(s_k)V_a(s_\ell)\} = Pr\{|s_k| < a, |s_\ell| < a\}.$$

Assume $\ell > k$ and note that $s_\ell - s_k$ is independent of s_k. Moreover, the density function of $s_\ell - s_k$ is clearly $\rho_{\ell-k}(x)$. We thus have

$$Pr\{|s_k| < a, |s_\ell| < a\} = \iint_\Omega \rho_k(x)\, \rho_{\ell-k}(y)dxdy,$$

where Ω is the region defined by the inequalities $-a < x < a$, $-a < x + y < a$. The area of this region is $4a^2$ and we have

$$\Pr\{|s_k| < a, |s_\ell| < a\} \sim \frac{4a^2}{k^{1/\alpha}(\ell-k)^{1/\alpha}} \rho^2(0).$$

It is now seen formally (and can be easily justified in a rigorous manner) that

$$(2.5) \qquad \lim_{n\to\infty} E\left\{\left(\frac{M_n(a)}{2a\,\rho(0)n^{1-1/\alpha}}\right)^2\right\} = 2! \iint_{0\le t_1\le t_2\le 1} \frac{dt_1 dt_2}{t_1^{1/\alpha}(t_2-t_1)^{1/\alpha}}.$$

In general, we get

$$(2.6) \qquad \lim_{n\to\infty} E\left\{\left(\frac{M_n(a)}{2a\,\rho(o)m^{1-1/\alpha}}\right)^m\right\} = m! \int\cdots\int_{0\le t_1\le t_2\le\cdots\le t_m\le 1} \frac{dt_1 dt_2\ldots dt_m}{t_1^{1/\alpha}(t_2-t_1)^{1/\alpha}\cdots(t_m-t_{m-1})^{1/\alpha}}$$

The integral on the right hand side of (2.6) can be easily reduced to beta functions obtaining

$$\int\cdots\int_{0\le t_1\le\cdots\le t_m\le 1} \frac{dt_1\ldots dt_m}{t_1^{1/\alpha}(t_2-t_1)^{1/\alpha}\cdots(t_m-t_{m-1})^{1/\alpha}}$$

$$= \int_0^1 \frac{d\xi}{\xi^{\frac{m}{\alpha}-(m-1)}} \prod_{k=1}^{m-1} \int_0^1 \frac{d\xi}{\xi^{\frac{k}{\alpha}-(k-1)}(1-\xi)^{1/\alpha}}$$

$$= \frac{\Gamma(m(1-\frac{1}{\alpha}))}{\Gamma(1+m(1-\frac{1}{\alpha}))}.$$

It now follows that the moment generating function of

$$(2.7) \qquad \frac{M_n(a)}{2a\,\rho(o)\,\Gamma(1-\frac{1}{\alpha})n^{1-\frac{1}{\alpha}}}$$

approaches, as $n\to\infty$, the function

$$(2.8) \qquad \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(1+m(1-\frac{1}{\alpha}))}.$$

It was recently shown by Pollard [3] that (2.8) is the moment generating function of the density function

$$(2.9)\qquad \phi(t) = \frac{1}{\pi(1-\frac{1}{\alpha})} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} \sin \pi(1-\frac{1}{\alpha})k\, \Gamma(1+k(1-\frac{1}{\alpha}))t^{k-1}$$

and thus

$$(2.10)\qquad \lim_{n\to\infty} \Pr\{M_n(a) < \frac{2ax}{\alpha \sin\frac{\pi}{\alpha}}\} = \int_0^x \phi(t)dt.$$

Use has been made of the facts that

$$\rho(0) = \frac{1}{\pi}\int_0^\infty e^{-\xi^\alpha} d\xi = \frac{1}{\pi\alpha}\Gamma(\frac{1}{\alpha}) \text{ and } \Gamma(\frac{1}{\alpha})\Gamma(1-\frac{1}{\alpha}) = \frac{\pi}{\sin\frac{\pi}{\alpha}}.$$

The treatment of N_n proceeds along similar lines. It is more convenient to consider the number $\bar{N}_n$ of s_k's, $1 \leq k \leq n$, for which $s_k > 0$, and $s_{k+1} < 0$. For large n the statistical properties of N_n and $2\bar{N}_n$ are the same.

Let

$$\psi(u,v) = \begin{cases} 1 & \text{if } u > 0,\ v < 0, \\ 0 & \text{otherwise.}\end{cases}$$

We have

$$\bar{N}_n = \sum_{k=1}^{n-1} \psi(s_k, s_{k+1})$$

and

$$E\{\bar{N}_n\} = \sum_1^{n-1} \Pr\{s_k > 0,\ s_{k+1} < 0\}.$$

Clearly

$$(2.11)\qquad \Pr\{s_k > 0,\ s_{k+1} < 0\} = \int_0^\infty\int_0^\infty \rho_k(x_1)\rho(x_1+x_2)dx_1dx_2 \sim$$

$$\sim \frac{\rho(0)}{k^{1/\alpha}} \int_0^\infty\int_0^\infty \rho(x_1+x_2)dx_1dx_2.$$

Here we make use of the fact that for $1 < \alpha \leq 2$, $\rho(x_1+x_2)$ is integrable (as a function of two variables).

In fact

$$\int_0^\infty \int_0^\infty \rho(x_1+x_2)dx_1 dx_2 = \int_0^\infty \int_{x_1}^\infty \rho(x)dx\, dx_1 =$$

$$\int_0^\infty x_1 \rho(x_1)dx = \frac{1}{2} \int_{-\infty}^\infty |x| \rho(x)dx = \frac{\mu}{2}$$

and for $1 < \alpha \leq 2$ the first absolute moment of X_j is finite. Thus

$$E\{\overline{N}_n\} \sim \frac{\mu}{2} \rho(0) \int_0^1 \frac{dt_1}{t_1^{1/2}} n^{1-1/\alpha} .$$

To calculate the second moment of $\overline{N}_n$ we need

(2.12) $\Pr\{s_k > 0, s_{k+1} < 0, s_\ell > 0, s_{\ell+1} < 0\}$.

Assuming $\ell > k+1$ (if $\ell = k+1$ the above probability is 0) we can write (2.12) as the quadruple integral

$$\iiiint_\Omega \rho_k(x)\, \rho(y)\, \rho_{\ell-k-1}(z) \rho(u)dx\, dy\, dz\, du,$$

where Ω is defined by the inequalities $x > 0$, $x+y < 0$, $x+y+z > 0$, $x+y+z+u < 0$. By a simple change of variable we reduce this integral to the form

$$\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \rho_k(x_1)\, \rho(x_1+x_2)\, \rho_{\ell-k-1}(x_2+x_3) \rho(x_3+x_4)dx_1 dx_2 dx_3 dx_4$$

which for large k and $\ell-k-1$ can be replaced by the asymptotic expression

$$\rho^2(0) \left(\frac{\mu}{2}\right)^2 \frac{1}{k^{1/\alpha} (\ell-k-1)^{1/\alpha}} .$$

It follows that

$$\lim_{n \to \infty} E\left\{\left(\frac{2\bar{N}_n}{\mu\rho(0)n^{1-1/\alpha}}\right)^2\right\} = 2! \int\int_{1 \le t_1 \le t_2 \le 1} \frac{dt_1 dt_2}{t_1^{1/\alpha}(t_2-t_1)^{1/\alpha}} .$$

Higher moments can be calculated in a similar manner and we see that in the limit $n \to \infty$ the distributions of

$$\frac{M_n(a)}{2a\rho(0)n^{1-1/\alpha}} \quad \text{and} \quad \frac{2\bar{N}_n}{\mu\rho(0)n^{1-1/\alpha}}$$

are identical.

2. <u>The case</u> $\alpha < 1$.

So far the behaviour of $M_n(a)$ and N_n were essentially identical. For $\alpha < 1$ (and $\alpha = 1$) this is not the case anymore.

We have (as in Section 2) a

$$E\{M_n(a)\} = \sum_{k=1}^{n} \frac{1}{k^{1/\alpha}} \int_{-a}^{a} \left(\frac{x}{k^{1/\alpha}}\right) dx <$$

$$< 2a\,\rho(0) \sum_{1}^{\infty} \frac{1}{k^{1/\alpha}} < \infty$$

and since $M_n(a)$ is non-decreasing sequence of random variables whose expectations are bounded it follows that

$$\lim_{n \to \infty} M_n(a) = M_\infty(a)$$

exists and is finite, with probability 1. The statistical properties of N_n are much more interesting.

For a fixed c consider

$$\iint_c^\infty \rho_k(x_1)\,\rho(x_1+x_2)dx_1\,dx_2$$

$$= k^{-1/\alpha} \iint_c^\infty \rho(k^{-1/\alpha}x_1)\,\rho(x_1+x_2)dx_1dx_2 = k^{1/\alpha} \iint_{ck^{-1/\alpha}}^{\infty} \rho(x_1)\,\rho(k^{1/\alpha}(x_1+x_2))dx_1dx_2.$$

By a theorem of Pólya [4], as $x \longrightarrow \infty$,

$$\rho(x) \sim C(\alpha)x^{-1-\alpha}$$

where

$$C(\alpha) = \frac{\Gamma(\alpha+1)\sin\frac{\pi\alpha}{2}}{\pi}.$$

It follows that as $k \longrightarrow \infty$

$$k^{1/\alpha} \iint_{ck^{-1/\alpha}}^{\infty} \rho(x_1)\rho(k^{1/\alpha}(x_1+x_2))dx_1dx_2 \sim$$

$$\sim \frac{C(\alpha)}{k} \iint_0^{\infty} \frac{\rho(x_1)}{(x_1+x_2)^{1+\alpha}} dx_1dx_2 = \frac{D(\alpha)}{k}$$

where

$$D(\alpha) = \frac{C(\alpha)}{\alpha} \int_0^{\infty} \frac{\rho(x)}{x^{\alpha}} dx.$$

Thus

$$\sum_{k=1}^{n-1} \iint_c^{\infty} \rho_k(x_1)\rho(x_1+x_2)dx_1\, dx_2 \sim D(\alpha) \log n.$$

In particular, taking $c=0$ we obtain (as in Section 2)

$$E\{\bar{N}_n\} \sim D(\alpha) \log n.$$

Now we put

$$W(\ell,n,c) = \sum \underbrace{\int \overset{\infty}{\cdots} \int}_{2\ell}{}_{c} \rho_{k_1}(x_1)\rho(x_1+y_1)\rho_{k_2-k_1-1}(x_1+y_1+x_2)\rho(x_1+y_1+x_2+y_2)\cdots$$

$$\rho_{k_\ell-k_{\ell-1}-1}(x_1+\cdots+y_{\ell-1}+x_\ell)\rho(x_1+\cdots+y_{\ell-1}+x_\ell+y_\ell)dx_1\cdots dy_\ell$$

where in the summation we have $1 \leq k_i \leq n$ and $k_i+1 < k_{i+1}$ for $i=1,\ldots,$.

We assume that

(3.1) $$W(\ell,n,c) \sim (D(\alpha)\log n)^{\ell}.$$

Then it is easy to see that

$$W(\ell+1,n,c) = \sum_{1 \leq k_1 \leq n-2} \iint_0^{\infty} \rho_{k_1}(x_1)\rho(x_1+y_1)W(\ell, n-k_1-1, c+x_1+y_1)dx_1dy_1$$

$$\sim \sum_{1 \leq k_1 \leq n-2} \frac{D(\alpha)}{k_1}(D(\alpha)\log(n-k_1-1))^{\ell}$$

$$\sim (D(\alpha)\log n)^{\ell+1}.$$

Since (3.1) has been shown to be true for $\ell=1$, it is true for all ℓ by induction. In particular, taking $c=0$ we obtain

$$W(\ell,n,o) \sim (D(\alpha)\log n)^{\ell} .$$

From this and the multinomial theorem we conclude that

$$E(\bar{N}_n^{\ell}) \sim \ell!\, W(\ell,n,o) \sim \ell!(D(\alpha)\log n)^{\ell} .$$

Consequently,

$$(3.2) \qquad \lim_{n \to \infty} \Pr\left\{ \frac{N_n}{D(\alpha)\log n} \leq x \right\} = \int_0^x e^{-y}\,dy = 1-e^{-x}.$$

One should emphasize here the curious fact that statistical behaviour of N_n for all $\alpha < 1$ is essentially the same, the only difference is in the numerical factor $D(\alpha)$.

4. The case $\alpha = 1$.

Proceeding as in Section 2 we obtain

$$E\{M_n(a)\} \sim 2a\rho(o)\sum_1^n \frac{1}{k} \sim 2a\rho(o)\log n$$

$$E\{M_n^2(a)\} \sim (2a\rho(o))^2 2! \sum_{1 \leq k \leq \ell \leq n} \frac{1}{k(\ell-k)} \sim 2!(2a\rho(o)\log n)^2.$$

It is easy to see that, in general,

$$E\{M_n^m(a)\} \sim m!(2a\rho(o)\log n)^m$$

and consequently

$$(4.1) \qquad \lim_{n \to \infty} \Pr\left\{ \frac{\pi M_n(a)}{2a\log n} < x \right\} = \int_0^x e^{-y}\,dy.$$

Here we made use of the fact that

$$\rho(x) = \frac{1}{\pi}\frac{1}{1+x^2}$$

so that $\rho(0) = 1/\pi$.

To investigate the behaviour of N_n we proceed like in Section 3. Suppose $c \neq 0$.

Then

$$k^{\frac{1}{\alpha}} \iint_{ck^{-1}}^{\infty} \rho(x_1)\rho(k^{\frac{1}{\alpha}}(x_1+x_2))dx_1dx_2$$

$$\sim \frac{1}{\pi^2 k} \iint_{ck^{-1}}^{\infty} \frac{1}{1+x_1^2} \frac{1}{(x_1+x_2)^2} dx_1dx_2 = \frac{1}{\pi^2 k} \int_{ck^{-1}}^{\infty} \frac{1}{(1+x^2)(ck^{-1}+x)} dx$$

$$= \frac{1}{\pi^2 k} \left(\log \frac{x}{1+x^2}\right) \Bigg|_{ck^{-1}}^{\infty} \sim \frac{\log k}{\pi^2 k}.$$

Thus

$$\sum_{k=1}^{n-1} \iint_{c}^{\infty} \rho_k(x_1)\rho(x_1+x_2)dx_1dx_2 \sim \frac{1}{\pi^2}(\log n)^2.$$

It can be shown that this result remains true for $c=0$. From this point on the argument is exactly the same as in the case $\alpha < 1$. Thus we obtain

(4.2) $$\lim_{n\to\infty} \Pr\left\{\frac{N_n}{\pi^2(\log n)^2} < x\right\} = \int_0^x e^{-y}\,dy.$$

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AN INVARIANCE PRINCIPLE FOR CERTAIN PROBABILITY LIMIT THEOREMS*

By

MONROE D. DONSKER

1. Introduction.

We consider a sequence $S_1, S_2, S_3, \cdots$ of partial sums of independent, identically distributed random variables $u_1, u_2, u_3, \cdots$ each having mean 0 and standard deviation 1. One of the impacts of the central limit theorem which states

$$\lim_{n \to \infty} \text{prob}\{S_n < \alpha n^{1/2}\} = (2\pi)^{-1/2} \int_{-\infty}^{\alpha} \exp(-t^2/2)dt$$

is that the limiting distribution is independent of the original distribution of the random variables. With regard to limit theorems, we say the 'invariance principle' holds in a particular case if a limiting distribution exists and is independent of the distribution of the random variables involved. A method developed by Erdös and Kac [1] and [2] [1], for proving certain limit theorems consists of first proving that the invariance principle holds and then calculating the limiting distribution by choosing a convenient distribution for the variables.

By Wiener space we mean here the space C consisting of all continuous functions $x(t)$ defined on $0 \leq t \leq 1$ $(x(0) = 0)$ with Wiener measure imposed on C.[2] By the distribution function $\sigma(\alpha)$ of a function $F(x)$ defined on C we mean $\sigma(\alpha) = \text{prob}\{F(x) < \alpha\}$. The object of this paper is to show that if $\{y_n\}$ is a family of random variables such that y_n is a function of $S_1, S_2, \cdots, S_n$, then under very weak restrictions the limiting distribution of y_n is the same as the distribution of a re-

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(1) Numbers in brackets refer to the bibliography at the end of the paper.

(2) For the definition of Wiener measure used here see N. Wiener, Generalized Harmonic Analysis, Acta Math., vol. 55(1930), pp. 117-258, especially pages 214-234.
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lated functional on C, i.e., the limiting distribution of y_n is independent of the distribution of the u's. In [1] Erdős and Kac demonstrated the invariance for the cases $y_n = \max(S_1, S_2, \ldots, S_n)$, $\max(|S_1|, |S_2|, \ldots, |S_n|)$, $\sum_{i=1}^{n} S_i^2$, and $\sum_{i=1}^{n} |S_i|$. These are special cases of Theorem (4.4) of this paper where the corresponding functionals are respectively

$$\max_{0 \leq t \leq 1} x(t), \quad \max_{0 \leq t \leq 1} |x(t)|, \quad \int_0^1 x^2(t)dt, \quad \int_0^1 |x(t)|dt.$$

In [2] the invariance was demonstrated for $y_n = N_n$ = number of positive partial sums in the sequence $S_1, S_2, \ldots, S_n$. This is also a special case of (4.4) where the corresponding functional is $\int_0^1 \frac{1 + \operatorname{sgn} x(t)}{2} dt$. In [3] Fortet proved the invariance property for the case $y_n = \sum_{i=1}^{n} S_i^{2m}$ under the restriction that the $2m^{th}$ moments be finite. Theorem (4.4) generalizes this result since the theorem holds with restrictions on only the first two moments and Fortet's result follows from the theorem by considering the functional $\int_0^1 x^{2m}(t)dt$. In [4] Mark proved several limit theorems analogous to those proved by Erdős and Kac in [1] using their technique of first proving the invariance principle and then calculating for convenient variables. Again the invariance principle for each of these limit theorems follows from (4.4) by considering the appropriate functional.

2. The invariance principle for a particular limit theorem.

Let k be a fixed positive integer, α and β be the vectors $\alpha:(\alpha_1, \alpha_2, \ldots, \alpha_k)$, $\beta:(\beta_1, \beta_2, \ldots, \beta_k)$ and define

$$n_i = \left[\frac{in}{k}\right] \qquad (i=0,1,2,\ldots,k).$$

Let E_n be the subset of the n-dimensional space $(u_1, u_2, \ldots, u_n)$ such that for all

$j=1,2,\ldots,k$ [1]

(2.1) $$(2n)^{1/2}\alpha_j \le S_i \le (2n)^{1/2}\beta_j \qquad n_{j-1} < i \le n_j$$

and let E be the subset of C such that for all $j=1,2,\ldots,k$

$$\alpha_j \le x(t) \le \beta_j \qquad t \in I_{kj}$$

where I_{kj} is the interval $\frac{j-1}{k} < t \le \frac{j}{k}$. We prove the following Theorem.[2]

(2.2) $$\lim_{n\to\infty} P\{E_n\} = P\{E\}.$$

Let $E_{n,r}$ be the subset of $(u_1,u_2,\ldots,u_n)$ such that the inequalities (2.1) are satisfied for $i=1,2,\ldots,r-1$ but are not satisfied for $i=r$. We see immediately that

(2.3) $$1-P\{E_n\} = \sum_{r=1}^{n} P\{E_{n,r}\}.$$

Let ν be any positive integer, $\varepsilon > 0$ be assigned, and consider the set of integers defined by

$$n_{j,p} = \left[\frac{(j-1)n}{k} + \frac{pn}{k\nu}\right] \qquad j=1,2,\ldots,k;\ p=0,1,\ldots,\nu .$$

For $n_{j,p} < r \le n_{j,p+1}$, we write

(2.4) $$P\{E_{n,r}\} = P\{E_{n,r}\}P\{|S_{n_{j,p+1}}-S_r| \ge \varepsilon\,(2n)^{1/2}\}+P\{E_{n,r}\}P\{|S_{n_{j,p+1}}-S_r| < \varepsilon(2n)^{1/2}\}.$$

Since the u's have mean 0 and standard deviation 1, we have from Tchebychef's inequality

(2.5) $$P\{|S_{n_{j,p+1}}-S_r| \ge \varepsilon(2n)^{1/2}\} \le 1/2\ \varepsilon^2 k\nu ,$$

---

(1) The normalizing factor 2n appears here rather than n so that we may later use the notation of Wiener integrals which seems to be convenient.

(2) Here and subsequently P{ } means the probability of the event or measure of the set defined by the braces. If the set is in C, this means Wiener measure.

which shows that the sum of the first terms on the right in (2.4) is at most $1/2\varepsilon^2\nu k$. Let F_n be the subset of $(u_1,u_2,\dots,u_n)$ such that for all $j=1,2,\dots,k$

$$(2n)^{1/2}\alpha_j \le S_{n_{j,p}} \le (2n)^{1/2}\beta_j \qquad (p=0,1,\dots,\nu),$$

and let $F_{n,\varepsilon}$ be the set defined by the same inequalities but with α_j and β_j replaced by $\alpha_j+\varepsilon$ and $\beta_j-\varepsilon$. The set $E_{n,r}\cap\{|S_{n_{j,p+1}} - S_r| < \varepsilon(2n)^{1/2}\}$ is obviously contained in the complement of $F_{n,\varepsilon}$ and hence the sum of the second terms on the right in (2.4) is at most $1-P\{F_{n,\varepsilon}\}$. On the other hand, $E_n\subset F_n$ and hence we get from (2.3) and (2.4)

$$P\{F_{n,\varepsilon}\} - 1/2\,\varepsilon^2 k\nu \le P\{E_n\} \le P\{F_n\}. \tag{2.6}$$

Let D_ν be the subset of C such that for all $j=1,2,\dots,k$

$$\alpha_j \le x\left(\frac{(j-1)\nu+p}{k\nu}\right) \le \beta_j \qquad (p=1,2,\dots,\nu)$$

and let $D_{\nu,\varepsilon}$ be the set defined by the same inequalities but with α_j and β_j replaced by $\alpha_j+\varepsilon$ and $\beta_j-\varepsilon$. It is a consequence of the multidimensional central limit theorem that

$$\lim_{n\to\infty} P\{F_{n,\varepsilon}\} = P\{D_{\nu,\varepsilon}\}$$

(2.7)

$$\lim_{n\to\infty} P\{F_n\} = P\{D_\nu\}\ .$$

In (2.6) if we hold ε and ν fixed and let $n\to\infty$ we get using (2.7)

$$P\{D_{\nu,\varepsilon}\} - 1/2\,\varepsilon^2 k\nu \le \liminf_{n\to\infty} P\{E_n\} \le \limsup_{n\to\infty} P\{E_n\} \le P\{D_\nu\}. \tag{2.8}$$

As $\nu\to\infty$ the sets D_ν form a sequence of measurable sets whose limit is E so that $P\{D_\nu\}\to P\{E\}$. Similarly $P\{D_{\nu,\varepsilon}\}\to P\{E_\varepsilon\}$ where E_ε is defined as E except that α_j and β_j are again replaced by $\alpha_j+\varepsilon$ and $\beta_j-\varepsilon$. Thus, if in (2.8) we first let $\nu\to\infty$ and then $\varepsilon\to 0$ we obtain the desired theorem.

For all $i=1,2,\dots,n$ define

$$x_n(t;w_1,w_2,\ldots,w_n) = \begin{cases} w_i & \text{for } \frac{i-1}{n} < t \leq \frac{i}{n} \\ w_1 & \text{for } t=0. \end{cases}$$

Let R_n be the subset of $(u_1,u_2,\ldots,u_n)$ such that for all $j=1,2,\ldots,k$

$$\alpha_j \leq x_n(t;S_1^*,S_2^*,\ldots,S_n^*) \leq \beta_j \quad \text{for } t\in I_{k,j} ,$$

where $S_i^* = S_i/(2n)^{1/2}$. By an argument analogous to the proof of Theorem (2.2) and using (2.2) it can be shown that

(2.9) $$\lim_{n\to\infty} P\{R_n\} = P\{E\}.$$

For the purposes of the next section it will be convenient to write (2.9) in a different but equivalent way. For each $j=1,2,\ldots,k$, let

$$p_j = \sup_{t\in I_{k,j}} x(t) \qquad q_j = \inf_{t\in I_{k,j}} x(t)$$

$$p_j^{(n)} = \sup_{t\in I_{k,j}} x_n(t;S_1^*,S_2^*,\ldots S_n^*) \quad q_j^{(n)} = \inf_{t\in I_{kj}} x_n(t;S_1^*,S_2^*,\ldots,S_n^*).$$

Thus p_j and q_j are each functionals on C and $p_j^{(n)}$ and $q_j^{(n)}$ are functions defined on $(u_1,u_2,\ldots,u_n)$. If we denote the characteristic function of a set A by χ_A, denote by B the 2k-dimensional interval

$$-\infty < \tau_i \leq \beta_i$$
$$\alpha_i \leq \tau_{k+i} < +\infty \qquad (i=1,2,\ldots,k),$$

and let $\phi(u)$ be the distribution function common to the u's, then (2.9) can be written[(1)]

(2.10) $$\lim_{n\to\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \chi_B[p_1^{(n)},p_2^{(n)},\ldots,p_k^{(n)};q_1^{(n)},q_2^{(n)},\ldots,q_k^{(n)}]d\phi(u_1)d\phi(u_2)\ldots d\phi(u_n)$$

---

(1) The notation $\int_C^{\omega} F[x]d_{\omega}x$ indicates the Wiener integral of $F[x]$ over C.

$$= \int_c^{\omega} \chi_B[p_1,p_2,\cdots,p_k;q_1,q_2,\cdots,q_k]d_\omega x.$$

By suitably combining blocks of the form B, we can obtain (2.10) for a general interval in the space $(\tau_1, \tau_2, \ldots, \tau_{2k})$. Using standard arguments of the Riemann approximation type and the fact that the random variable $(p_1,p_2,\ldots,p_k;q_1,q_2,\ldots,q_k)$ has a density function[(1)], we can obtain the

(2.11) Theorem. Let $f(\tau_1, \tau_2, \ldots, \tau_{2k})$ be Borel measurable, bounded on the entire space $(\tau_1, \tau_2, \ldots, \tau_{2k})$, and Riemann integrable on every finite 2k-dimensional interval. Then,

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(p_1^{(n)}, p_2^{(n)}, \ldots, p_k^{(n)}; q_1^{(n)}, q_2^{(n)}, \ldots, q_k^{(n)}) d\phi(u_1) \ldots d\phi(u_n)$$

$$= \int_c^{\omega} f(p_1,p_2,\ldots,p_k;q_1,q_2,\ldots,q_k)d_\omega x$$

3. An approximation theorem.

Let R be the space of functions $g(t)$ which are continuous except possibly for a finite number of finite jumps on $0 \leq t \leq 1$. Let Q be the set of functionals defined on R of the form $f(\tau_1, \tau_2, \ldots, \tau_{2k})$ where

$$\tau_j = \sup_{t \in I_{kj}} g(t) \qquad \tau_{k+j} = \inf_{t \in I_{kj}} g(t) \qquad (j=1,2,\ldots,k)$$

and where $f(\tau_1, \tau_2, \ldots, \tau_{2k})$ satisfies the assumptions of theorem (2.11). We prove the following

(3.1) Theorem.

Let $F(g)$ be bounded and uniformly continuous in the uniform topology on R. Then, there exists a pair of sequences of functionals $\{F_k^*(g)\}$ and $\{F_k^{**}(g)\}$ all belonging to Q such that for each k and all g in R

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(1) For the case k=2 see Fürth, Ann. D. Phys., 53:1917, p. 177.

$$(3.2) \qquad F_k^{**}(g) \leq F(g) \leq F_k^{*}(g)$$

and such that

$$(3.3) \qquad \lim_{k \to \infty} \int_C^{\omega} (F_k^{*}[x] - F_k^{**}[x]) d_{\omega}x = 0.$$

Let

$$g_k^{*}(t) = \sup_{u \in I_{kj}} g(u) \qquad \text{when } t \in I_{kj}$$

$$g_k^{**}(t) = \inf_{u \in I_{kj}} g(u) \qquad \text{when } t \in I_{kj}$$

and let M_g be the set of functions $h(t)$ in R such that $g_k^{**}(t) \leq h(t) \leq g_k^{*}(t)$.

We then define

$$F_k^{*}(g) = \sup_{h \in M_g} F(h)$$

$$F_k^{**}(g) = \inf_{h \in M_g} F(h)$$

We show that the sequences $\{F_k^{*}(g)\}$ and $\{F_k^{**}(g)\}$ have the desired properties. It is obvious from their definition that $F_k^{*}(g)$ and $F_k^{**}(g)$ satisfy (3.2). Since $F_k^{*}(g)$ and $F_k^{**}(g)$ are both bounded by the bound of $F(g)$, it is sufficient in order to establish (3.3) to show the integrand in (3.3) tends to zero for each fixed $x(t)$ in C. This follows easily, however, from the fact that $x(t)$ is uniformly continuous on $0 \leq t \leq 1$ and from the assumption that $F(g)$ is uniformly continuous in the uniform topology.

From the definition of $F_k^{*}(g)$ and $F_k^{**}(g)$ it is clear that they are functions of the form $f(\tau_1, \tau_2, \ldots, \tau_{2k})$ where

$$\tau_j = \sup_{t \in I_{kj}} g(t) \qquad \tau_{k+j} = \inf_{t \in I_{kj}} g(t) \qquad (j=1,2,\ldots,k).$$

To show that $f(\tau_1, \tau_2, \ldots, \tau_{2k})$ has the required properties it is sufficient to show that $f(\tau_1, \tau_2, \ldots, \tau_{2k})$ is bounded and continuous in all its 2k variables on a Jordan measurable region and zero on its complement. Let Γ be the subset of

$(\tau_1, \tau_2, \ldots, \tau_{2k})$ such that $\tau_j \geq \tau_{k+j}$ for $j=1,2,\ldots,k$. We note that Γ is both Borel and Jordan measurable. To show that $F_k^*(g)$ is in Q we define

$$f_1(\tau_1, \tau_2, \ldots, \tau_{2k}) = \sup F(h)$$

where the supremum is taken as $h(t)$ varies over all functions in R satisfying for all $j=1,2,\ldots,k$

$$\tau_{k+j} \leq h(t) < \tau_j \qquad \text{when } t \in I_{kj},$$

provided this set of $h(t)$ is not empty. If it is empty, let $f_1(\tau_1, \tau_2, \ldots, \tau_{2k})=0$. The set of $(\tau_1, \tau_2, \ldots, \tau_{2k})$ for which the set of $h(t)$ is empty is clearly the complement of Γ. Now the function $f_1(\tau_1, \tau_2, \ldots, \tau_{2k})$ which is defined over the whole space $(\tau_1, \tau_2, \ldots, \tau_{2k})$ is clearly equal to $F_k^*(g)$ wherever this latter function is defined. Hence it will be sufficient to show that $f_1(\tau_1, \tau_2, \ldots, \tau_{2k})$ is continuous on Γ in the sense that if $\varepsilon > 0$ is given and $(\tau_1, \tau_2, \ldots, \tau_{2k}) \in \Gamma$, then there exists a $\delta > 0$ such that

$$(3.4) \qquad |f_1(\tau_1, \tau_2, \ldots, \tau_{2k}) - f_1(\tilde{\tau}_1, \tilde{\tau}_2, \ldots, \tilde{\tau}_{2k})| < \varepsilon$$

whenever $|\tau_j - \tilde{\tau}_j| < \delta$ for $j=1,2,\ldots,2k$ and $(\tilde{\tau}_1, \tilde{\tau}_2, \ldots, \tilde{\tau}_{2k}) \in \Gamma$. To show this, choose $g(t) \in R$ such that for $j=1,2,\ldots,k$

$$\tau_j = \sup_{t \in I_{kj}} g(t) \qquad \text{and} \qquad \tau_{k+j} = \inf_{t \in I_{kj}} g(t)$$

and for any other element of R, call it $\tilde{g}(t)$, let

$$\tilde{\tau}_j = \sup_{t \in I_{kj}} \tilde{g}(t) \qquad \text{and} \qquad \tilde{\tau}_{k+j} = \inf_{t \in I_{kj}} g(t).$$

By assumption, there exists a $\Delta > 0$ such that for any two functions $h_1(t)$ and $h_2(t)$ for which $\sup_{0 \leq t \leq 1} |h_1(t) - h_2(t)| < \Delta$ we have $|F(h_1) - F(h_2)| < \varepsilon/2$. Let $M_{\tilde{g}}$ denote the set of functions $h(t)$ in R such that $\tilde{g}_k^{**} \leq h(t) \leq \tilde{g}_k^*$. In order to show (3.4) we see from the definition of $F_k^*(g)$ that we must show there exists a $\delta > 0$

such that when $g(t)$ is such that $|\tau_j - \tilde{\tau}_j| < \delta$ for $j=1,2,\dots,2k$ then

(3.5) $$\left| \sup_{h \in M_g} F(h) - \sup_{h \in M_{\tilde{g}}} F(h) \right| < \varepsilon.$$

There exists a function $h_1(t)$ in M_g such that

(3.6) $$\sup_{h \in M_g} F(h) - F(h_1) < \varepsilon/2.$$

Let $h_1'(t) = h_1(t)$ at all points t where

$\inf_{t \in I_{kj}} \tilde{g}(t) < h_1(t) < \sup_{t \in I_{kj}} \tilde{g}(t)$ and $h_1'(t) = \inf_{t \in I_{kj}} \tilde{g}(t)$ if $h_1(t)$

$\leq \inf_{t \in I_{kj}} g(t)$ and $h_1'(t) = \sup_{t \in I_{kj}} \tilde{g}(t)$ if $h_1(t) \geq \sup_{t \in I_{kj}} \tilde{g}(t)$.

Thus $h_1'(t)$ as defined is an element of $M_{\tilde{g}}$ and hence

(3.7) $$F(h_1'(t)) \leq \sup_{h \in M_{\tilde{g}}} F(h).$$

If we let $\delta = \Delta$, then from the definition of $h_1'(t)$ we have

$$|h_1(t) - h_1'(t)| < \Delta \text{ and hence}$$

(3.8) $$F(h_1) < \varepsilon/2 + F(h_1').$$

From (3.6), (3.7), and (3.8) we get

(3.9) $$\sup_{h \in M_g} F(h) - \sup_{h \in M_{\tilde{g}}} F(h) < \varepsilon.$$

By a symmetrical argument we can show that

$$\sup_{h \in M_{\tilde{g}}} F(h) - \sup_{h \in M_g} F(h) < \varepsilon$$

and hence the desired inequality (3.5).

A similar argument would work for $f_2(\tau_1, \tau_2, \dots, \tau_{2k})$, where $f_2(\tau_1, \tau_2, \dots, \tau_{2k})$ is defined on the space $(\tau_1, \tau_2, \dots, \tau_{2k})$ and is equal to $F_k^{**}(g)$ wherever this latter function is defined. This completes the proof of theorem (4.1).

4. The general invariance principle.

From (2.11) and (3.1) we see that if F(g) is bounded and uniformly continuous in the uniform topology on R, then

$$(4.1) \qquad \lim_{n\longrightarrow} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} F[x_n(t{:}S_1^*,S_2^*,\ldots,S_n^*)]\, d\phi(u_1)d\phi(u_2)\ldots d\phi(u_n) = \int_C^{\omega} F[x]d_{\omega}x.$$

Let V be the class of bounded functionals F(g) on R which are continuous in the uniform topology at almost all points (Wiener measure) of C. We state the following

(4.2) Theorem[(1)]

A necessary and sufficient condition that a functional F(g) be in V is that given an $\varepsilon > 0$, there exist two functionals $F_1(g)$ and $F_2(g)$ both bounded and uniformly continuous in the uniform topology on R satisfying the conditions

$$F_1(g) \leq F(g) \leq F_2(g)$$

$$\int_C^{\omega} [F_2(x) - F_1(x)]\, d_{\omega}x < \varepsilon.$$

From (4.1) and (4.2) we see that if F(g) is bounded on R and is continuous in the uniform topology at almost all points of C, then

$$(4.3) \qquad \lim_{n\longrightarrow\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} F[x_n(t;S_1^*,S_2^*,\ldots,S_n^*)]\, d\phi(u_1)\ldots d\phi(u_n) = \int_C^{\omega} F[x]d_{\omega}x$$

Although (4.3) is stated for a real valued function on R, it is clear that (4.3) holds also for a complex valued function on R. Thus in (4.3) if we replace F(g) by $e^{izF(g)}$ we still satisfy the conditions that imply (4.3) even if now F(g) is unbounded and hence using the continuity theorem for characteristic functions and letting $\sigma(\alpha)$ denote the distribution function of F(x) i.e. $\sigma(\alpha) = P\{F(x) < \alpha\}$, we get the

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(1) The proof of this theorem follows closely the proof of the analogous classical necessary and sufficient condition for Riemann integrability.

(4.4) Theorem.

If $F(g)$ is defined on R and is continuous (uniform topology) at almost all points (Wiener measure) of C, then at every point of continuity of $\sigma(\alpha)$ we have

$$\lim_{n \to \infty} P\{ F[x_n(t;S_1^*,S_2^*,\ldots,S_n^*)] < \alpha \} = \sigma(\alpha).$$

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